Discussion upon the Stationarity and the Causality of a Time Series Using the Phase Diagram Representation

Răzvan SCRIPCARU, Angela DIGULESCU, Leontin TUȚĂ, and Alexandru ȘERBĂNESCU

***Abstract*—This paper discusses the properties of stationarity and causality for given time series. These characteristics are highly important for nonlinear time series acquired in real application. The link between the stationarity and causality of a signal is not unequivocally obtained. In the same time, causality of a time series and its phase diagram representation are strongly related and, the latter represents an alternative for the classical test, given that it is a non-parametric data-driven tool which presents many advantages for the dynamic time series highlighting the causal/ chaotic behavior of a process.**

***Index Terms*—causality, linear methods, phase diagram, polar coordinates.**

I. Introduction

In the field of research, it is useful to obtain approximately the same results for exactly the same process in order to best characterize the studied process. Thus, the reproducibility of the experiment is essential and is characterized by its stationarity.

A form of stationarity is given by the process that has constants all relevant parameters during a recording. In other words, for the same process that occurs under the same conditions, the parameters extracted from the measurements should be the same when the experiment is repeated.

When the process is a probabilistic one, the probability distribution functions should be time independent. For a deterministic process, the effect of the inputs must follow the same rule during the measurements.

For real applications, the signal source is often unavailable and stationarity is not easily determined. Therefore, a second form of stationarity has been developed and is based on the time series recorded from the process measurement. In this case, it should be noted that this form of stationarity is different from the first one, because there are systems that are stationary for infinite long time series, but are non-stationary for short periods of time.

The causality test and the phase diagram representation are presented along with the necessary requirements for the desired information to be obtained unequivocally.

Therefore, a signal is stationary, if its probabilities of evolution of states are independent of time in the observation interval [1]. This means that the relevant parameters that describe the dynamics of the process are contained in the measurement. If the signal contains irregular bursts, then the signal is non-stationary even if the parameters remain constant.

Non-stationary signals are very common in natural phenomena description. Their study is very advantageous because it provides information about systems that are governed by complex mathematical equations.

A stationarity test is required to determine whether a signal is stationary or not. First, the time series must be much longer than the evolution duration of the longest evolution parameter. If the measurement is performed for a shorter period than this parameter, then the time series must be considered non-stationary.

The paper is organized as follows: Section II presents the theoretical aspects of the proposed paper; Section III compares different types of signals and discusses their characteristics. Finally, Section IV draws some conclusions and suggests some future applications.

II. Theoretical Aspects

A. Stationarity of a signal

The simplest way to check whether a signal is stationary or not is to check its stationarity in the broad sense: the moments of first and second order, such as mean and signal variance. If they are time independent, then the signal is stationary, otherwise it is not.

Therefore, in the case of stationarity, the fundamental properties of the process must not change when they are checked for several segments of the recorded signal.

For example, for a given time series, $X = \{x[1], x[2], ..., x[N]\}$, the mean, m_x, and the standard deviation, σ for several segments of the signal must be calculated:

$$m_x = \frac{\sum_{n=1}^{N} x[n]}{N} \quad (1)$$

$$\sigma = \sqrt{\frac{\sum_{n=1}^{N} \left(x[n] - m_x\right)^2}{N-1}} \quad (2)$$

$$\sigma_{err} = \frac{\sigma}{\sqrt{N}} \quad (3)$$

While (1) describes the signal mean and (2) shows its

R. SCRIPCARU is with the Military Technical Academy “Ferdinand I”, Bucharest. Romania. (e-mail: razvan.scripcaru@mta.ro)

A. DIGULESCU is with the Military Technical Academy “Ferdinand I”, Bucharest. Romania. (e-mail: angela.digulescu@mta.ro)

L. TUȚĂ is with the Military Technical Academy “Ferdinand I”, Bucharest. Romania. (e-mail: leontin.tuta@mta.ro)

A. ȘERBĂNESCU is with the Military Technical Academy “Ferdinand I”, Bucharest. Romania. (e-mail: alexandru.serbanescu@mta.ro)

Digital Object Identifier 10.32754/JMT.2019.2.02

standard deviation (how far away from the mean are the signal samples), (3) is known as the standard error when estimating the average of the uncorrelated numbers with Gaussian distribution.

If the average remains the same within the fluctuations given by the standard error and variance, then the signal is stationary, otherwise it is not. The latter is often the case of natural phenomena.

It is worth mentioning that the above equations are only estimators of the respective quantities, because the results are available from the time series and not from the probability density function of the process.

In addition, information about the time dependence of two measures can be fulfilled by autocorrelation functions:

$$r_{xx}[k] = \frac{\sum_{i=1}^{N-k}(x_i - m_x)(x_{i+k} - m_x)}{\sum_{i=1}^{N}(x_i - m_x)^2} \quad (4)$$

(4) shows the similarity between the signal and a delayed copy of itself.

B. *Causality test*

Another way for characterization of a signal is dynamic system theory approach [3, 4]. One approach is the causality of the system [3] and it can be detected by the recurrence method.

Given the time series $X = \{x[1], x[2], ..., x[N]\}$, the radius r, and indices i and j such that $i, j = \overline{1, N}$, then:

$$\text{if } |x[i] - x[j]| < r \Rightarrow R{+}{+}$$
$$\text{and if } |x[i+1] - x[j+1]| < r \Rightarrow CR{+}{+} \quad (5)$$
$$\text{then } iz_x = \frac{CR}{R}$$

where R is the recurrence of the series for the given r, CR is the consecutive recurrence which is also counted when both condition are simultaneous accomplished; the ratio between the recurrence and the consecutive recurrence, hereinafter called consecutive isometry, iz_x, represents a measure of causality.

In the causality test of a system, the representation according to the radius is used, the recurrences being checked for different values of the radius.

C. *Phase diagram representation*

Inspired from the dynamic system theory [4], a second approach is the phase diagram representation.

Given a time series $X = \{x[1], x[2], ..., x[N]\}$, by representing a vector whose projections are the time series samples, we can discover certain information that could not be observed from the representation of the samples according to the temporal parameter. This representation of the values of the time series according to its delayed values is called phase diagram representation [4]. It can be considered, at the same time, the successive representation of the states of the process that generated that time series, in the phase plane, a representation that bears the name of attractor [5].

The attractor of the given time series can be reconstructed using an encapsulation parameter m and a time delay d, using the following relation [5]:

$$\vec{v_i} = (x[i], x[i+d], x[i+2d], \ldots, x[i+(m-1)d]) \quad (6)$$

where $i = 1, 2, \ldots, M$, and $M = N - (m-1)d$.

Figure 1 shows such a representation.

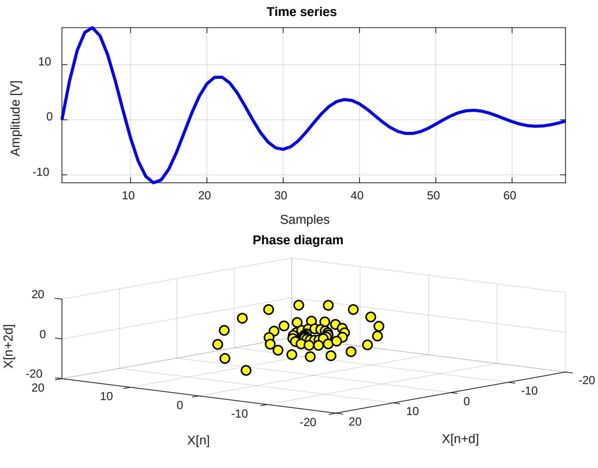


Figure 1. Time series (up) and its phase diagram representation (bottom)

III. Results and Discussion

The most important question, however, remains: which methods are more effective in characterizing a process, linear or non-linear ones?

A. *Chaotic vs causal*

For this, three different types of signals are considered and an attempt is made to establish an answer. We will study a pure random process, a chaotic one and a non-stationary causal one. The random process and the deterministic process may have a chaotic or casual behavior, depending on the value of parameter g in the equation describing the process:

$$A(t+1) = A(t) + g \cdot \sin(A(t)) \quad (7)$$

The initial solution for both series is: $A(1) = 1$.

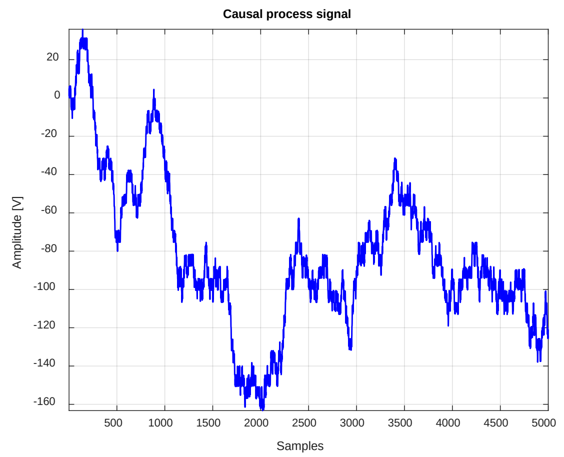


Figure 2. Time series corresponding to a causal process $(g = 4.7)$

From the point of view of the linear methods, will be highlighted the estimated probability distributions of the signals presented in Figure 2 and Figure 3 These represent, in reality, the histogram of the signal and, as can be seen from Figure 4, a defining characteristic of the signal cannot be obtained. It is worth to mention that the estimators from (1)-(3) are based on this parameter.

 Digital Object Identifier 10.32754/JMT.2019.2.02

In the problem of characterization of the signal type (deterministic/chaotic/random) the linear methods seem to reach their limits quickly, because the process is influenced by components with nonlinear dependence.

As we can see in Figure 4 our two cases offer a completely different behavior: the chaotic signal has most values as extremes of the histogram, and the deterministic signal has a similar distribution to the Gaussian one, but with a non-zero average (the distribution is shifted to the left of zero).

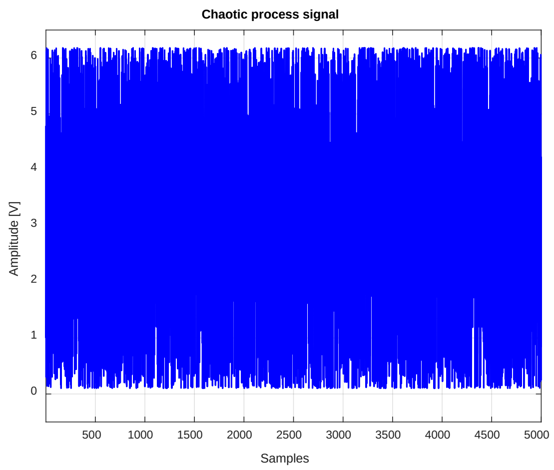


Figure 3. Time series corresponding to a chaotic process $(g = 4.5)$

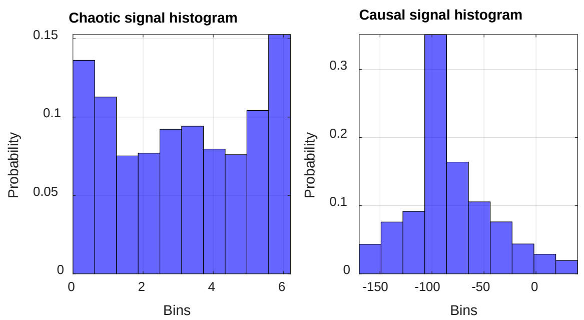


Figure 4. Histogram of chaotic (left) and causal signal (right)

Moreover, if the distribution of the deterministic signal is compared with that of a white Gaussian noise, it can be seen that the similarity between them is very high.

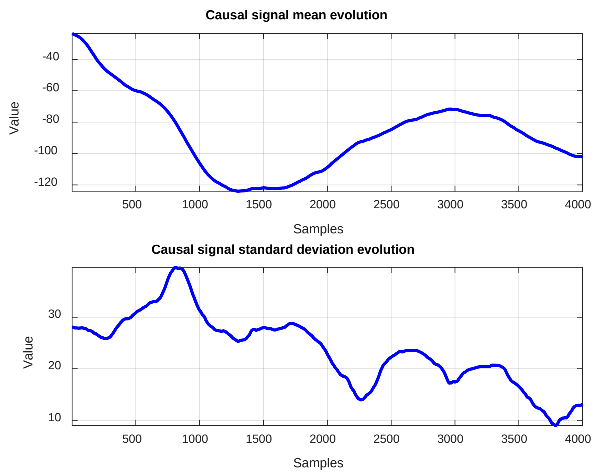


Figure 5. Causal signal mean and variance evolution computed for a sliding window $w = 1000$

For the chaotic signal the autocorrelation function is linear increasing/decreasing related to the perfectly overlapped point, while for the causal signal the autocorrelation fuction has greater values, non-linear, which highlight the dependence between the delayed samples.

From Figure 5 and Figure 6, it can be noticed that although the deterministic signal is causal its mean and its standard deviation vary a lot in time. Whereas for the chaotic signal the same parameters are quasi-constant, therefore this process is stationary.

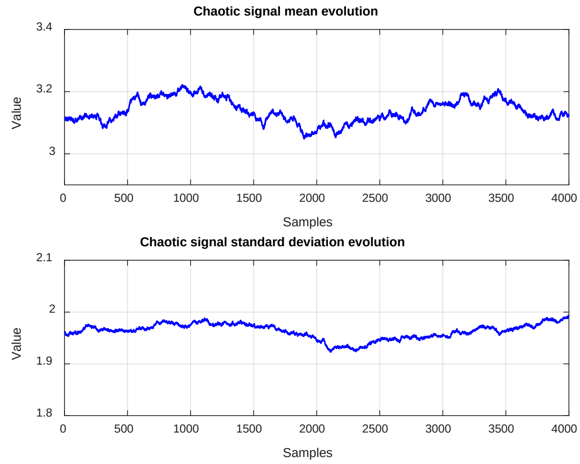


Figure 6. Chaotic signal mean and variance evolution computed for a sliding window $w = 1000$

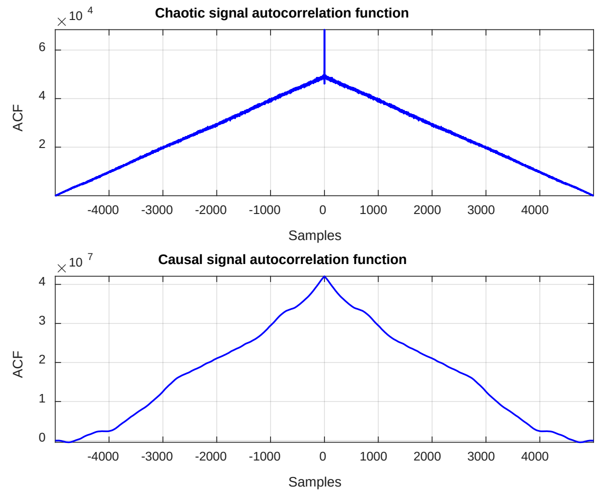


Figure 7. Time series autocorrelation function

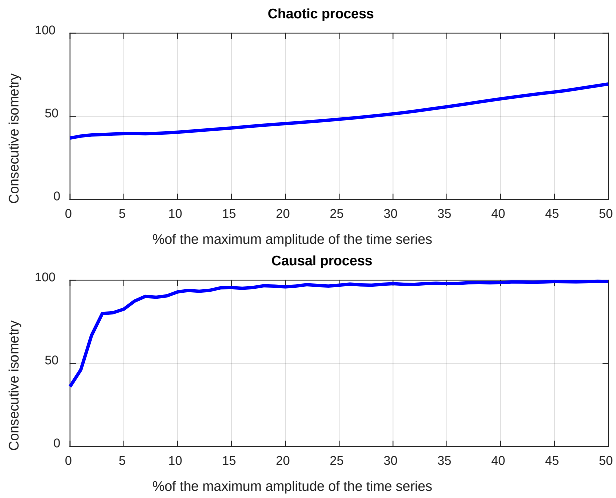


Figure 8. The percentage of consecutive recurrence represented by the radius for the studied processes: chaotic and causal

For the causal time series the consecutive isometry is greater than 90% for a radius of 10% highlighting a strong correlation for the recursive time series, but for the chaotic process the consecutive isometry does not exceed 45% for the same radius, meaning that the relationship between consecutive samples is occasional.

As shown in Figure 9 the phase diagram for the chaotic process has a uniform parallelepiped distribution which means that the position vectors have not attractor. On the other hand, after analyzing the phase diagram representation for the casual signal, we observe that the position vectors evolve around an ellipsoid which forms the attractor of the process. Hereby the causality of a signal is highlighted in the phase diagram true the presence of an attractor.

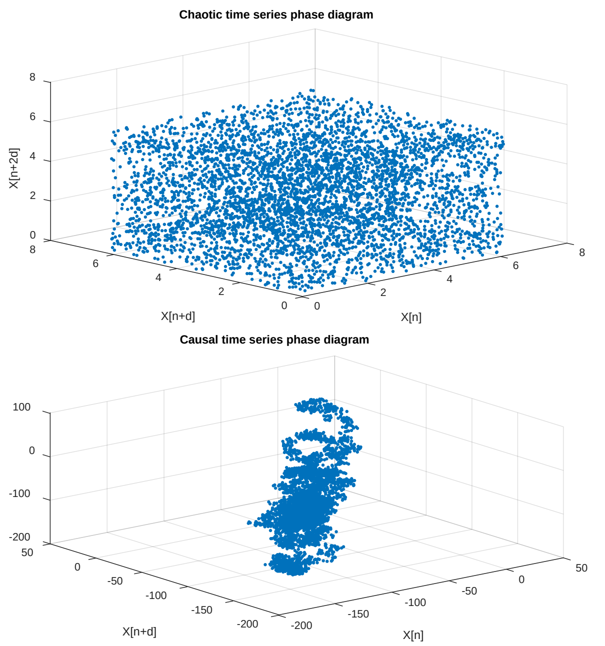


Figure 9. The phase diagram representation for the chaotic and causal signal, computed using a time delay $d = 50$

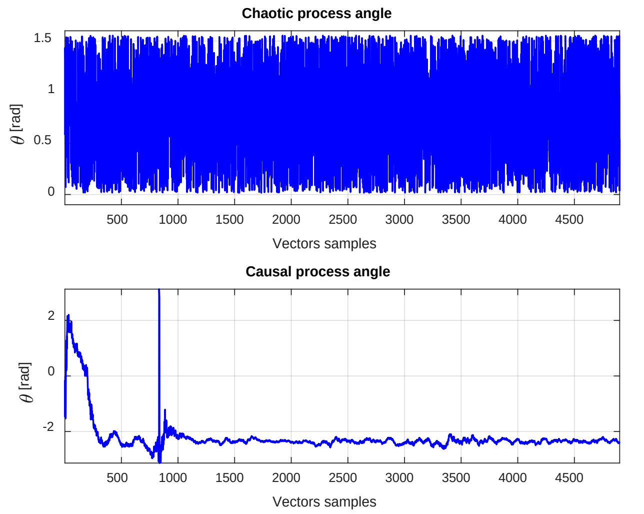


Figure 10. The angles of position vectors that form phase diagram representation for the studied process: chaotic and causal

For the chaotic signal, Figure 10 and Figure 11 emphasize the independence between the angles of the position vectors from the phase diagram, while for the causal process the same representation shows a strong interdependence between the angles of the position vectors that form the phase diagram attractor, highlighting that some parameters are triggering the process.

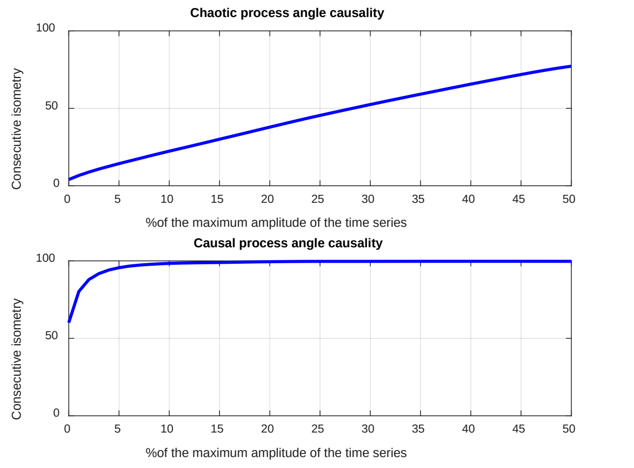


Figure 11. The percentage of consecutive recurrence represented by the radius for angle of the 3D position vectors of the phase diagram trajectory corresponding to the studied processes: chaotic and causal

B. Logistic function

The algorithms presented above were also applied to the logistic function which is describe by the equation:

$$x[i+1] = k \cdot x[i](1 - x[i]) \quad (8)$$

with the initial solution: $x[1] = 0.2$.

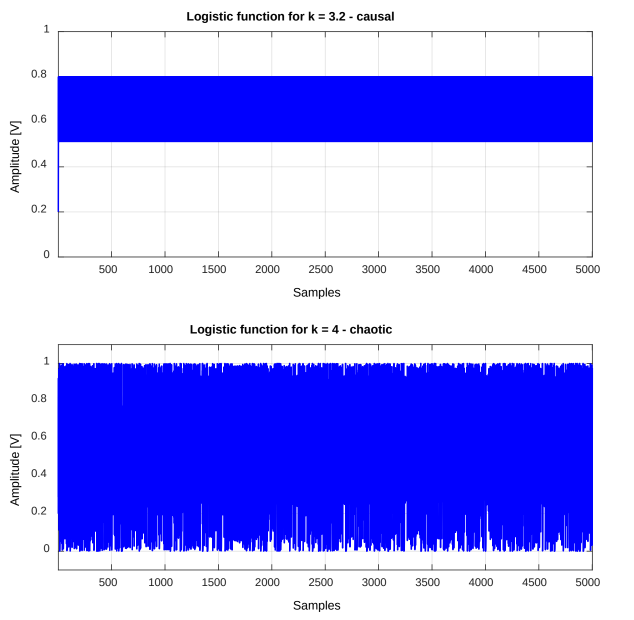


Figure 12. Logistic function time series corresponding to a causal process (up) and to a chaotic process (bottom)

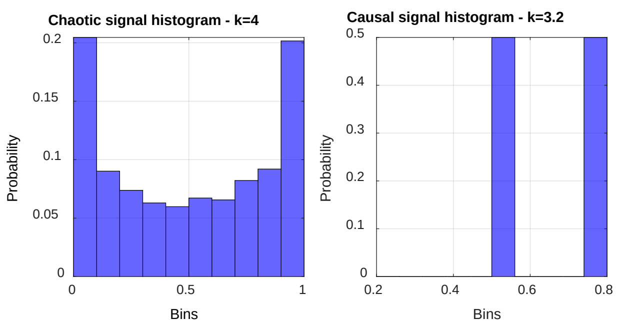


Figure 13. Logistic function histogram corresponding to a chaotic process and to a causal process

Figure 13 displays a quasi-uniform distribution for the chaotic process and a two value oscillating system for the causal case.

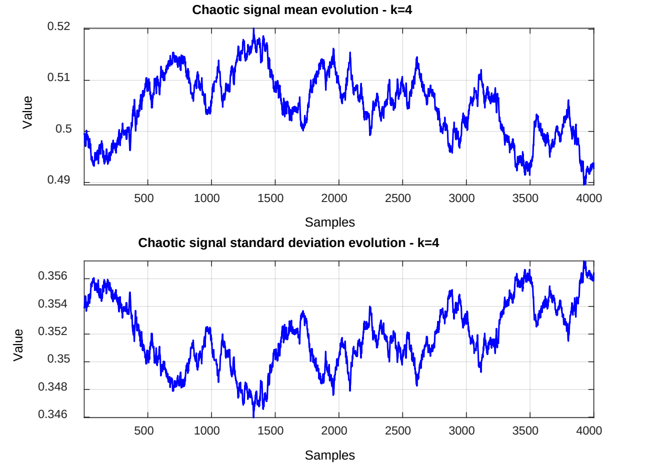


Figure 14. Chaotic signal mean and variance evolution computed for a sliding window $w = 1000$

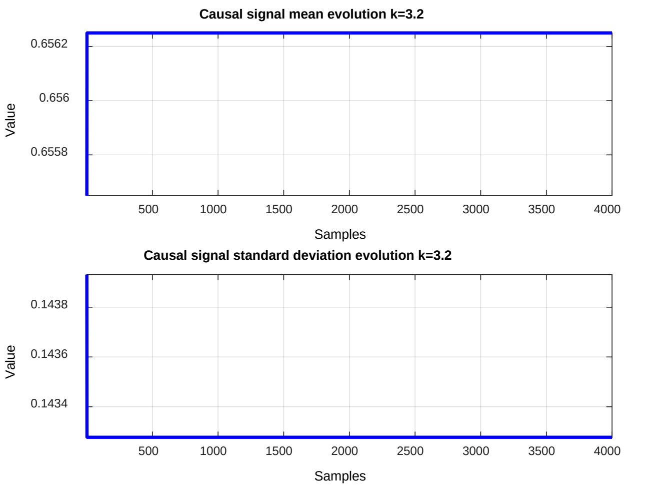


Figure 15. Causal signal mean and variance evolution computed for a sliding window $w = 1000$

Both cases for the logistic function have the parameters (almost) constant in time which means that they are stationary (Figure 14 and Figure 15).

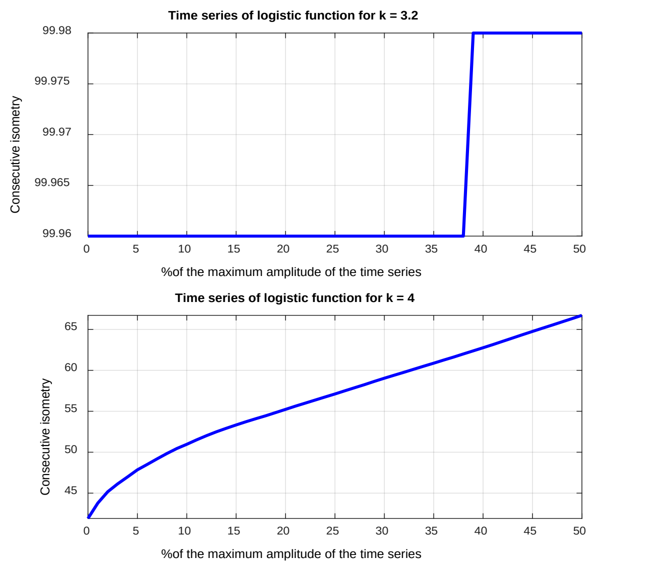


Figure 16. Causality test for the logistic function corresponding to a causal process (up) and to a chaotic process (bottom)

Figure 16 highlights that the logistic function for $k = 3.2$ is around 100% independent of the radius, while the logistic function for $k = 4$ has an evolution similar to the chaotic process from the previous section.

From the phase diagram representation it can be noticed that the logistic function for $k = 3.2$ presents just 3 states where it evolves, resulting in a continuous oscillation (Figure 17). In the same time, the logistic function for $k = 4$ has a parabolic shape which forms the attractor of this process.

Figure 18 shows the evolution of the angle of the position vectors which maintain the behavior from the phase diagram trajectory. For $k = 3.2$ the values of the angle just oscillate between two values while the chaotic evolution is present for $k = 4$.

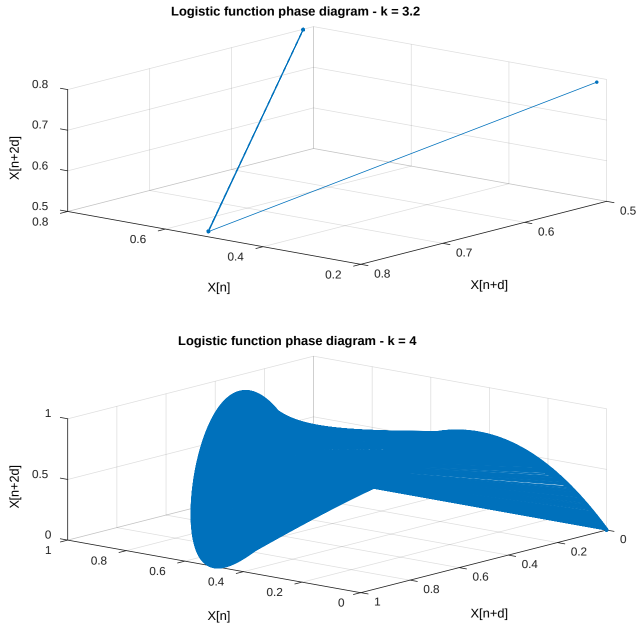


Figure 17. The phase diagram representation for the chaotic and causal signal, computed using a time delay $d = 1$

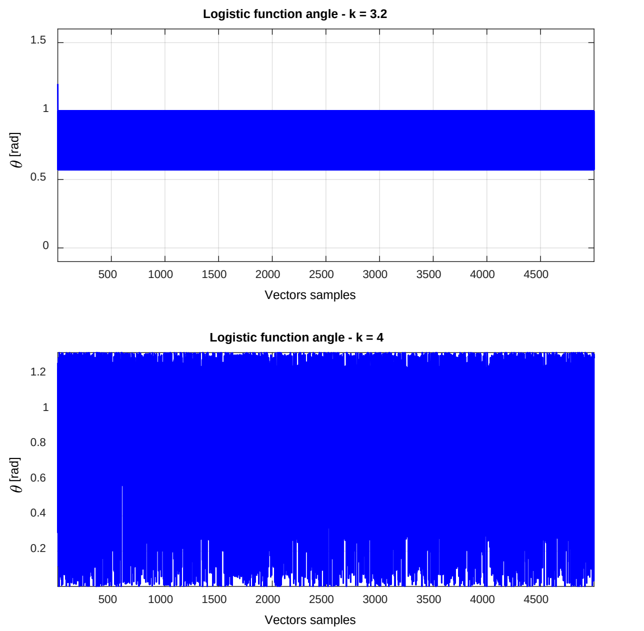
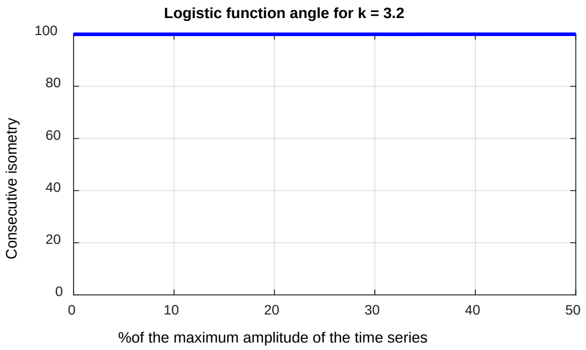


Figure 18. The angles of position vectors that form phase diagram representation for the logistic function

The causality test for the position vector shows a significant improvement for the causal/ chaotic behavior, as Figure 19 shows, the consecutive recurrence being ideal for $k = 3.2$ meanwhile for $k = 4$ this behavior is strongly chaotic and different from a random process (whose the causality test is equivalent to the first bisector representation [3])



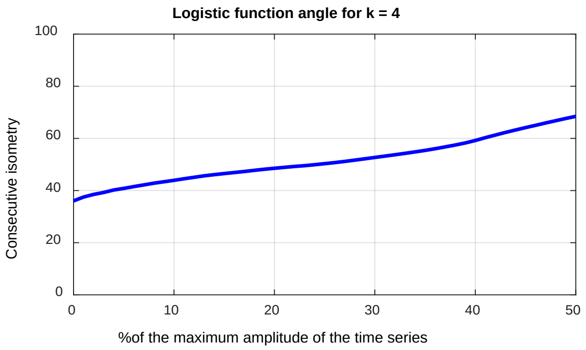


Figure 19. The percentage of consecutive recurrence represented by the radius for angle of the 3D position vectors of the phase diagram trajectory corresponding to the studied processes: $k = 3.2$ and $k = 4$

IV. CONCLUSION

This paper presents the concepts of stationarity and causality applied on famous cases known in the dynamic systems theory [1, 3, 4].

The properties of stationarity and causality are described and applied on a chaotic and deterministic (causal) signal highlighting that the studied properties are not directly related.

Moreover, he property of causality has been transferred to the phase diagram concept and by applying it to the logistic function for two familiar cases (oscillating system and chaotic system), we have proved that, by applying the causality test on the vector position angle, the system's property is better emphasized.

Further work foresees the application of these methods on signals coming from natural systems.

REFERENCES

[1] H. Kantz, T. Schreiber, "Nonlinear Time Series Analysis", University Press, Cambridge, 1997.

[2] A. Serbanescu, O. Stanasila, F. M. Birleanu, "Nonlinear analysis of time series", Military Technical Academy Publishing, Bucharest, 2011.

[3] H. Sabelli, A. Sugerman, L. Kovacevic, L. Kauffman, L. Carlson-Sabelli, M. Patel, J. Konecki, "Bios Data Analyzer", Nonlinear Dynamics Psychol Life Sci. 2005 Oct; 9(4):505-38.

[4] J. Eckmann, S. Kamphorst and D. Ruelle, "Recurrence Plots of Dynamical Systems", Europhysics Letters 440 (EPL), pp. 973-977, 1987, DOI: 10.1209/0295-5075/4/9/004.

[5] C. Ioana, A. Digulescu, A. Serbanescu, I. Candel, F.-M. Birleanu, "Recent advances in non-stationary signal processing based on Recurrence Plot Analysis", 5th International Symposium on Recurrence Plots Chicago, august 14-16 aug., 2013. Published in Ed. Springer 2014: Translational Recurrences. From Mathematical Theory Real-World Applications, Springer Proceedings in Mathematics & Statistics, 2014.