On the Choice of the Weight Function and its Parameters in the Element Free Galerkin Method

Silvia MARZAVAN

***Abstract*—Our paper is focused on a subject having a great importance when Element Free Galerkin (EFG) method is used. Namely, it is about the choosing of some charateristic parameters of the kernel function. Practically, the EFG method user has to decide upon some aspects like which kernel function is going to be used, then which is the most fitted dimension of the domain support, which is the influence of the internodal distance for a better accuracy and others. This paper is based on the theoretical fundamentals of the EFG method and on our practical experience in using of the method. Our research presents some comments and recommendations accompanied by examples and processing of some results coming from practice. The author considers that despite all the progress made in the development of the EFG method, its use in the analysis of mechanical structures needs a good experience regarding some aspects on which the indications are few, or vague, or missing. The numerical examples in this work, are performing by using original programs in Matlab for solving simple problems. We consider that our conclusions are very useful not only for the beginners in using of EFGM but for those who want to know better and especially why it is better to use an one or other option.**

***Index Terms*—kernel or weight function, element-free Galerkin, meshfree, meshless.**

I. Introduction

Not many years ago, a new generation of computational methods — meshfree and meshless methods appeared. Any numerical method has its own limits or in other words no numerical method is a perfect one. This point of view could be the reason of appearing new and new numerical methods, although the Finite Element Method (FEM) existed and it is known and widely used in almost all engineering problems. Among the new numerical methods for the analysis of mechanical structures or fluid mechanics problems, those methods named meshless or meshfree methods are the best known and used, being in continuum development. Methods like Smoothed Particle Method (SPH) and Element Free Galerkin (EFG) method are most used; they are a viable alternative to the FEM in many circumstances and by many reasons. A pure meshless method is SPH method, but many authors accepted the meshfree concept to be more general, including meshless concept. So, a mesh-free or meshless method (G.R. Liu, 2002) is a method which establishes an algebraic equation system for the whole problem domain without using a predefined mesh for the domain discretization.

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These two meshfree methods use kernel functions, so many aspects regarding kernel functions used in EFG method are similar with the kernel functions used in SPH method. But between SPH method and EFG method many differences exist; so, our research, presented in this paper, is referring only to EFG method.

The element-free Galerkin method is almost identical to the finite element method, but the essential difference consists in the different ways of interpolation. The discretization of the governing equations by the element-free Galerkin method requires moving least-square approximants, which are made up of three components: a weight function associated with each node, a monomial basis, and a set of nonconstant coefficients. Therefore there is no need for element and element connectivity data as with the FEM.

The meshfree methods use a set of nodes scattered within the problem domain. These nodes don't represent a discretization of the problem domain. Such nodes do not form a mesh and they are called field nodes. Meshless consists in subdividing the structure in nodes. At each node, we associate a weight (kernel) function and a shape function defined on a small domain, around the node. FEM uses only shape functions which are defined on what we call finite elements.

Some authors, to whom we join, distinguish between "interpolation" and "approximation". Interpolation refers to an evaluation procedure that reproduces the exact values of the approximated function at the nodes. All the other evaluation procedures that do not return nodal function values are called approximation. These concepts are illustrated in Fig. 1.

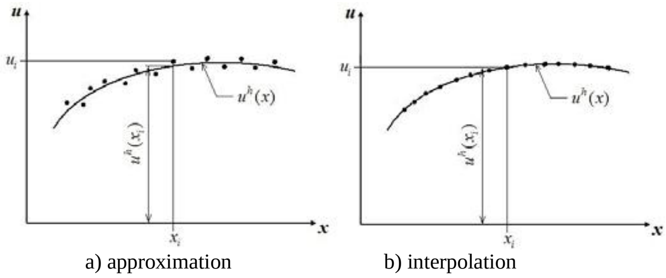

Figure 1. The evaluation by approximation a) and by interpolation b)

Both interpolation and approximation are used in meshfree methods. The standard FEM uses interpolation based on elements. The EFG method uses approximation.

II. Weight Functions

Many weight functions [1-3] have been designed, tested and used in the meshfree method; some of them are presented below.

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The cubic spline weight function:

$$w(r)=\begin{cases}\frac{2}{3}-4r^2+4r^3 & r\le 0.5\\ \frac{4}{3}-4r^2-\frac{4}{3}r^3 & 0.5<r\le 1\\ 0 & r>1\end{cases} \quad (1)$$

The quartic weight function:

$$w(r)=\begin{cases}1-6r^2+8r^3-3r^4 & r\le 1\\ 0 & r>1\end{cases}, \quad (2)$$

The new quartic weight function:

$$w(r)=\begin{cases}\frac{2}{3}-\frac{9}{2}r^2+\frac{19}{3}r^3-\frac{5}{2}r^4 & r\le 1\\ 0 & r>1\end{cases} \quad (3)$$

The cubic weight function:

$$w(r)=\begin{cases}1-3r^2+2r^3 & 0\le r\le 1\\ 0 & r>1\end{cases} \quad (4)$$

Other cubic weight function:

$$w(r)=\begin{cases}1-\frac{3}{2}r^2+\frac{1}{2}r^3 & 0\le r\le 1\\ 0 & r>1\end{cases} \quad (5)$$

The exponential weight function:

$$w(r)=\begin{cases}e^{-r^2} & r\le 1\\ 0 & r>1\end{cases} \quad (6)$$

The elliptical weight function:

$$w(r)=\begin{cases}2\sqrt{1-r^2} & 0\le r\le 1\\ 0 & r>1\end{cases} \quad (7)$$

The cosine weight function:

$$w(r)=\begin{cases}\cos\left(\frac{\pi r}{2}\right) & 0\le r\le 1\\ 0 & r>1\end{cases} \quad (8)$$

Lucy weight function:

$$w(r)=\begin{cases}(1+3r)\left(1-r^2\right)^3 & r\le 1\\ 0 & r>1\end{cases} \quad (9)$$

The cubic B-spline weight function (Monaghan and Lattanzio):

$$w(r)=\begin{cases}1-\frac{3}{2}r^2+\frac{3}{4}r^3 & r\le 1\\ \frac{1}{4}(2-r)^3 & 1<r\le 2\\ 0 & 2<r\end{cases} \quad (10)$$

Monaghan-Lattanzio cubic spline weight function:

$$w(r)=\begin{cases}\frac{2}{3}-r^2+\frac{1}{2}r^3 & 0\le r\le 1\\ \frac{1}{6}(2-r)^3 & 1<r<2\\ 0 & r\ge 2\end{cases} \quad (11)$$

Morris weight function, a quartic spline function:

$$w(r)=\begin{cases}\left(\frac{5}{2}+r\right)^4-5\left(\frac{3}{2}+r\right)^4+10\left(\frac{1}{2}+r\right)^4 & 0\le r\le\frac{1}{2}\\ \left(\frac{5}{2}-r\right)^4-5\left(\frac{3}{2}-r\right)^4 & \frac{1}{2}\le r<\frac{3}{2}\\ \left(\frac{5}{2}-r\right)^4 & \frac{3}{2}\le r\le\frac{5}{2}\\ 0 & \frac{5}{2}\le r\end{cases} \quad (12)$$

and a quintic spline function:

$$w(r)=\begin{cases}(3-r)^5-6(2-r)^5+15(1-r)^5 & 0\le r<1\\ (3-r)^5-6(2-r)^5 & 1\le r<2\\ (3-r)^5 & 2\le r<3\\ 0 & 3<r\end{cases} \quad (13)$$

Johnson quadratic weight function:

$$w(r)=\frac{3}{4}-\frac{3}{4}r+\frac{3}{16}r^2 \qquad 0\le r\le 2 \quad (14)$$

which is recommended for numerical simulation of high velocity impact problem.

Certainly, not all weight functions are presented above. Many others can be created using theoretical conditions for constructing weight functions. Looking at the weight functions presented above, we easily notice a first problem that arises for the user of a meshfree method: *what weight function to use?* We will give an answer below.

In all weight functions presented here, the variable r is a vector defining the relative distance between a point of interest and other points around it, by the relation (15)

$$(r)_I=\frac{\left\|x-x_I\right\|}{d_{mI}}, \quad (15)$$

where x_I is the coordinate of the point of interest, x is the coordinate of a neighboring point; d_{mI} is the size of the influence domain of node I. The size of the influence domain may be different from that of the support domain, but practically, often, no difference exists. It is calculated by the relation:

$$d_{mI}=d_{\mathrm{m}}\cdot c_I, \quad (16)$$

where d_{m} is a scaling parameter, with a recommended value between 2 and 4, for static analysis. Its value can be constant or variable from one node to another. Here is another problem: *what is the most appropriate value?* The answer follows.

In (16), c_I is a distance determined by the need to have enough neighbor points as the matrix not to be singular. All these parameters are presented in the Fig. 2, for the case of 1D problem.

The number of nodes (N_{nd}) on the d_{mI} dimension of the support domain can be calculated with the relation:

$$N_{nd}=\frac{d_{mI}}{d_{nn}}+1 \quad (17)$$

Based on (17) we can calculate the internodal distance (d_{nn}) to ensure a certain number of nodes on the dimension d_{mI} of the support domain

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$$d_{nn} = \frac{d_{mI}}{N_{nd} - 1} \quad (18)$$

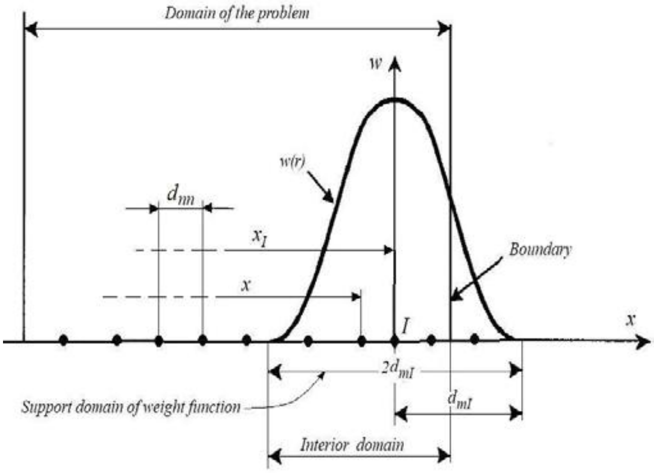

Figure 2. Parameters of the weight function (space 1D)

For example, looking at Fig. 3, let us consider the case of the new quartic weight function; we find that the support domain has the dimension $d_{mI} = 0.2$ and internodal distance $d_{nn} = 0.1$, resulting:

$$N_{nd} = \frac{0.2}{0.1} + 1 = 2 + 1 = 3 \quad (19)$$

Indeed, looking at the Fig. 3, for the new quartic weight function, we find that there are 3 nodes in the d_{mI} dimension of the support domain.

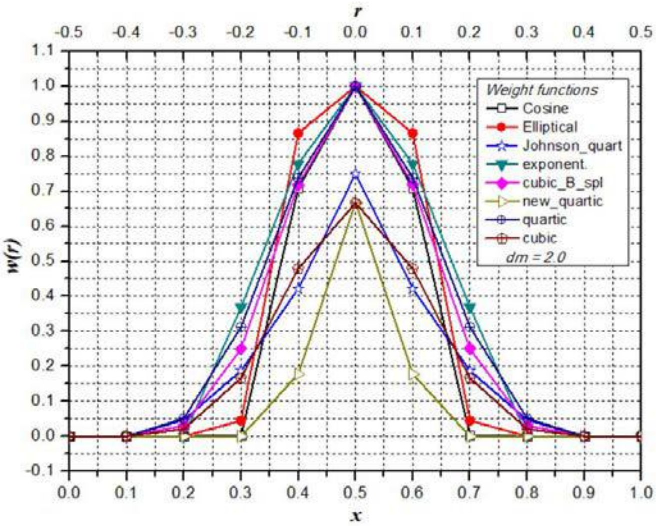

Figure 3. Graphical representation of weight functions

Practice shows that too many nodes in the support domain do not improve accuracy, and too few nodes lead to an irreversible matrix, so the solution cannot be found. The analysis of the curves presented in Fig. 3 shows us something else: the curvature for the new quartic weight functions and Johnson quartic function is different from the others. This fact shows us that the share of nodes in the vicinity of the node of interest is lower. The nodes in the immediate vicinity of the node of interest have the largest share [6], [7].

Such a situation is preferable for the description of phenomena with a strong local character, as it is the case of impact problems in general and high speed impact in particular [4], [5].

The same figure also shows us that some weight functions have an almost identical graphical representation. This fact shows us that in the calculation with the EFG method, the choice of one or the other of them has a less importance or does not matter.

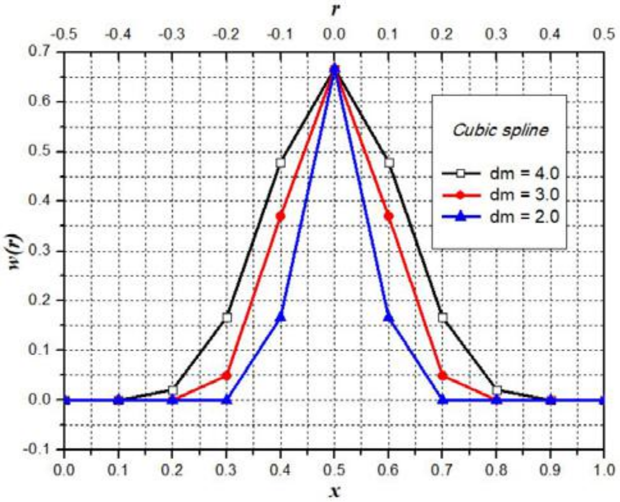

Figure 4. Cubic spline weight function with different support domain size

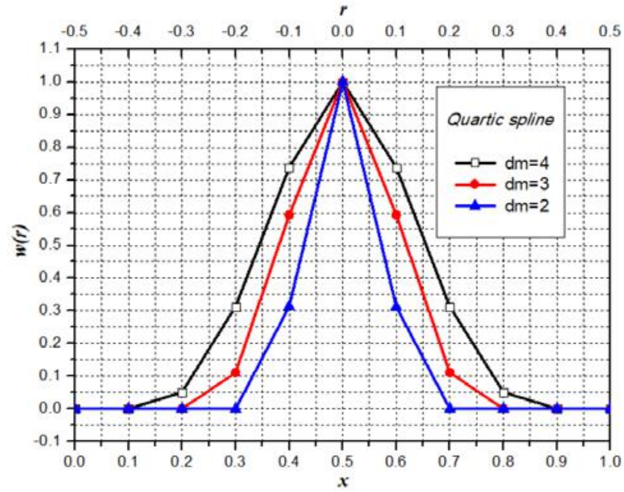

Figure 5. Quartic weight function with different support domain size

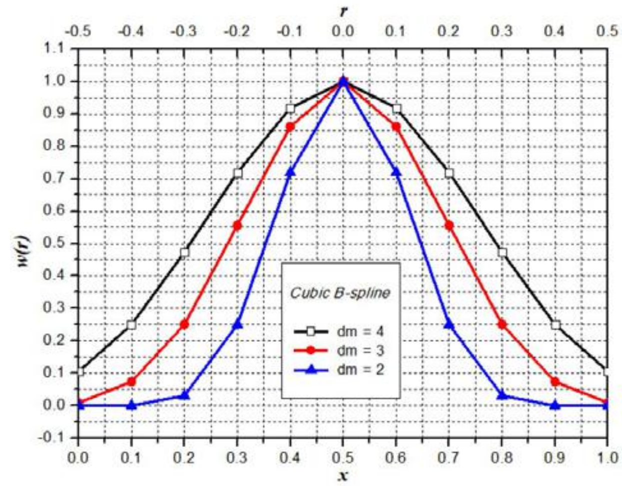

Figure 6. Cubic B-spline function with different support domain size

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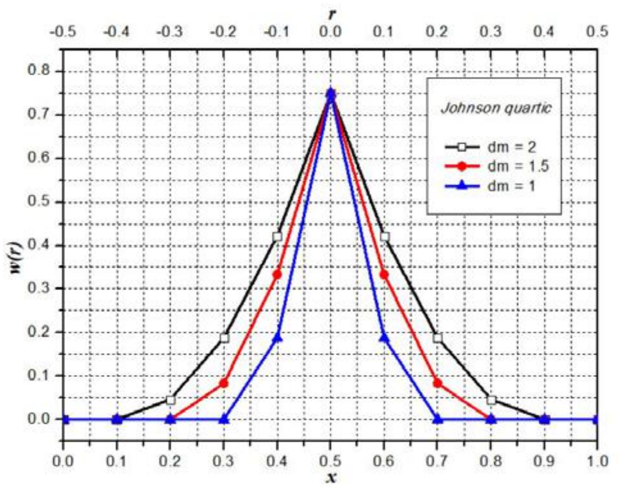

Figure 7. Johnson quartic spline function with different support domain size

Fig. 4-7 show some graphical representations of the most used weight functions, for different support domain size. These graphical representations show us the dependence d_m between the size of the support domain and the number of nodes contained in it. Too large size of the support domain can cause it to have a size comparable to or larger than the problem domain (Fig. 6); this observation is more pronounced in the case of the Johnson quartic weight function.

For this reason, in Fig. 7 the representation was made for lower values of the parameter d_m.

III. Theoretical Case Study

In order to substantiate some conclusions regarding the choice of the weight function and the parameter that defines the size of the support domain, the case study is presented below.

Let be a straight bar, axially stressed by a force uniformly distributed along the longitudinal axis, as seen in Fig. 8; for simplicity of calculation, the length of the bar is equal to unity, also the area of the cross section and Young's modulus have values equal to unity.

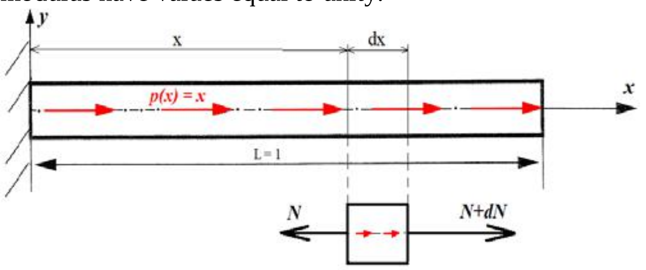

Figure 8. The analitical calculus model

For the above considered problem, when $L=1$, $A=1$ and $E=1$, we obtain the following analytical solution:

$$u(x)=\frac{x}{2}-\frac{x^3}{6} \tag{20}$$

For the numerical analysis with the EFG method, we can consider the numerical calculus model presented in Fig. 9.

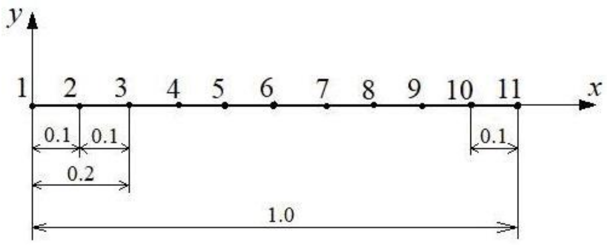

Figure 9. Calculus model for EFG method

The analytical solution (nodal displacements), made based on the relation (20) is presented in Table I, A), B).

TABLE I. ANALITICAL RESULTS, A)

x	0	0.1	0.2	0.3	0.4
Node	1	2	3	4	5
$u(x)$	0	0.050	0.099	0.146	0.189

TABLE I. ANALITICAL RESULTS, B)

x	0.5	0.6	0.7	0.8	0.9	1
Node	6	7	8	9	10	11
$u(x)$	0.229	0.264	0.293	0.315	0.329	0.333

Considering the nodal analytical values as parameters of the nodal displacements, we will successively use some weight functions, with different dimensions of the support domain to establish the values of the numerical calculus.
These will be compared with the values of the analytical results. Thus, some useful conclusions, for the practice with the EFG method, will be able to be established [8], [9].

A. *Cubic spline weight function*

The cubic spline weight function is one of the most used in EFG method. In the Fig. 10, the using of the weight function, for each node of the problem domain, is presented in a suggestive graphical way.

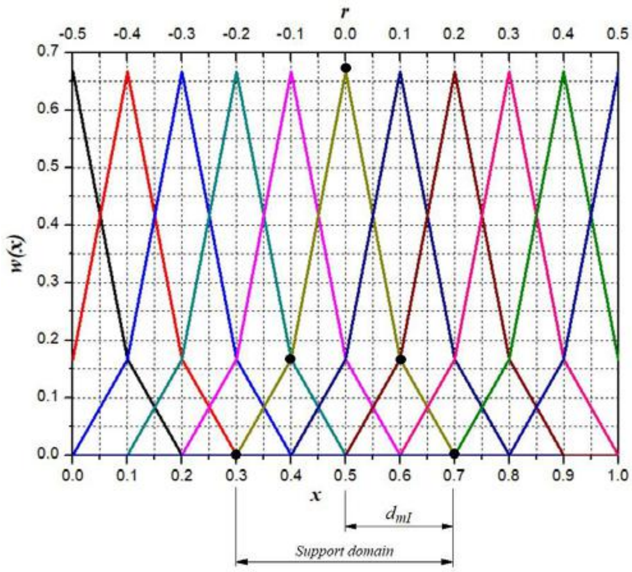

Figure 10. Successive use of the chosen weight function

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The results and the working way are shown in Fig. 11, in a graphical presentation. The calculus errors, for each node, compared to the analytical results, are also presented.

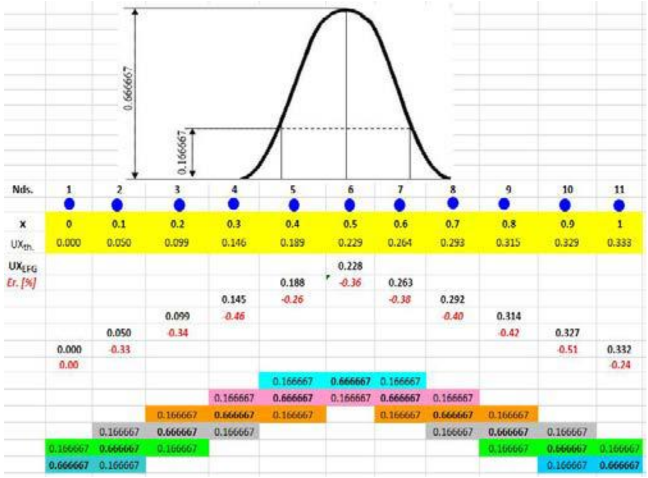

Figure 11. The using of the cubic weight function, $d_m = 2.0$

In Fig. 11 the results of the using of weight function are presented. The weight function was placed in each node, with its center on the node. The position, shown in Fig. 11, is the one corresponding to node 6. The result from this node is calculated as suggestively is shown in the figure:

$$UX_{EFG}^{(6)} = 0.166667 \cdot 0.189 + 0.666667 \cdot 0.229 + 0.166667 \cdot 0.264 \quad (21)$$

$$UX_{EFG}^{(6)} = 0.228167 \approx 0.228 \quad (22)$$

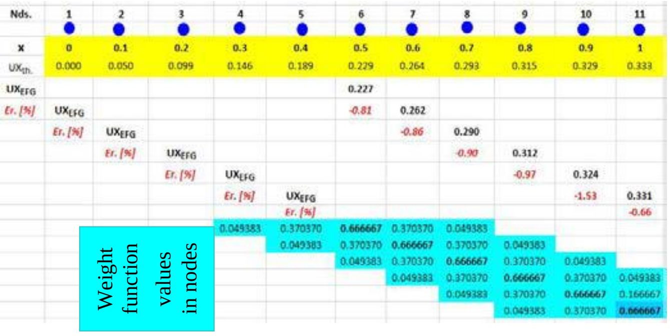

Figure 12. The using of the cubic weight function, $d_m = 3.0$

Using the same methodology, we proceeded to use the same weight function, but for a different values of the d_m parameter.

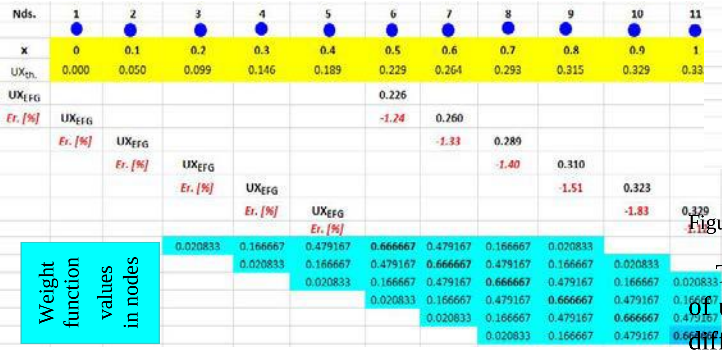

Figure 13. The using of the cubic weight function, $d_m = 4.0$

In Fig. 12 and 13, for a more concise presentation, the graphical presentation of the calculation is made only for the range of positive values of the vector r (only for the nodes 6, 7, 8, 9, 10, 11).

B. *Quartic Spline Weight Function*

Along with the cubic spline function, the quartic spline function is often invoked in structure analysis programs with the EFG method. In the following figures (Fig. 14-17) this function is tested, in the same way.

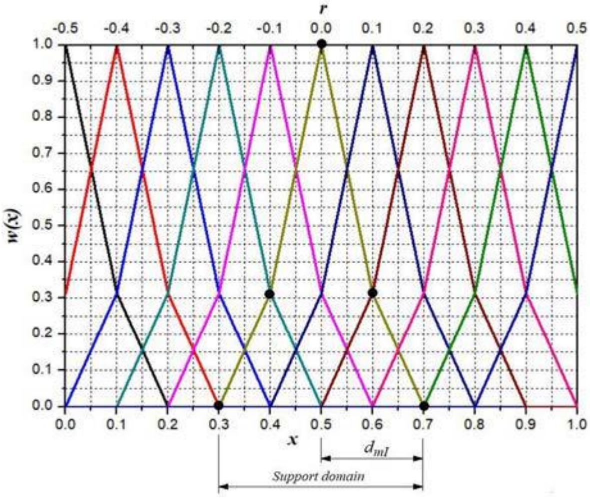

Figure 14. Successive use of the quartic weight function

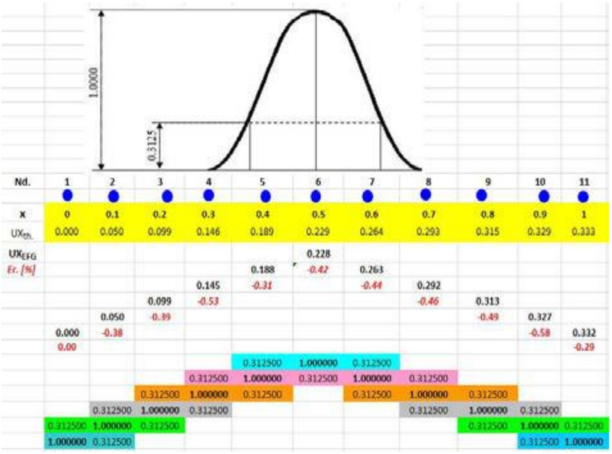

Figure 15. The using of the quartic weight function, $d_m = 2.0$

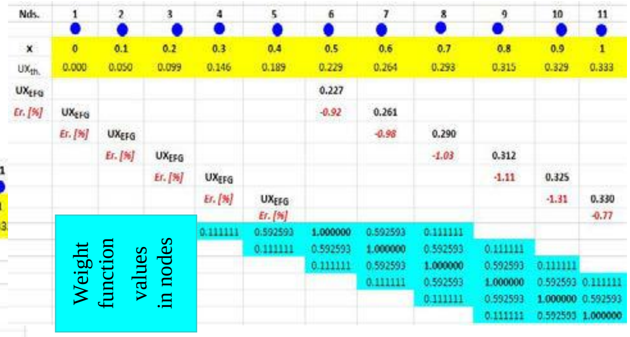

Figure 16. Quartic spline kernel function, $d_m = 3.0$

This subchapter presents, in a suggestive manner, the way of using the weight function for the considered case, for the different values (2...4) of the dimensional parameter d_m. The three values considered cover the range of values indicated in the literature [10], [11]. From the analysis of the highlighted errors, it results that the most appropriate value of the dimensional parameter d_m is 2.0, both for the cubic spline function and for the quartic spline function.

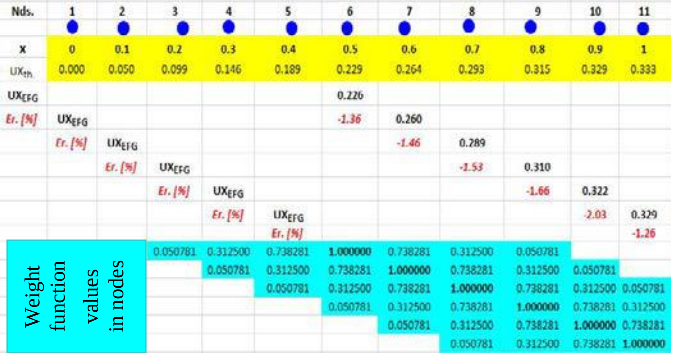

Figure 17. Quartic spline kernel function, $d_m = 4.0$

From the analyzed example, there is a slight advantage as a recommendation to be used, for the cubic spline function, with the value 2.0 of the parameter d_m.

Our research also shows that the use of other weighting functions could lead to significantly higher errors; this observation justifies the wide use of cubic and quartic spline functions in the analysis with the EFG method.

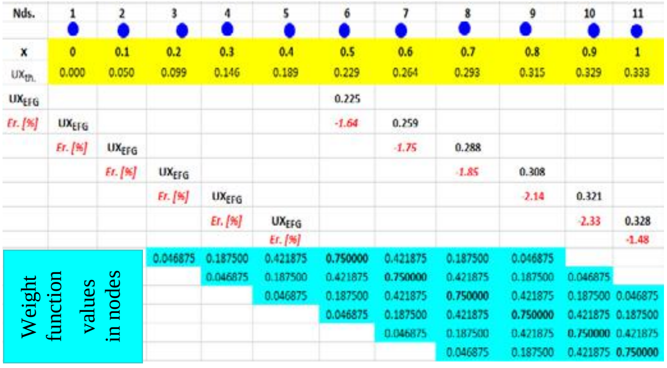

Figure 18. Johnson quartic function, $d_m = 2.0$

Fig. 18-20 show the results of using the Johnson quartic function. To obtain results comparable to those of using cubic and quartic spline functions, the dimensional parameter d_m was used in the range of 1.0 to 2.0. For this weight function, for $d_m = 1.0$, the results are the best (Fig. 20). This aspect could be explained by the alure of this weght function presented in the Fig. 7. The share of nodes, even in the support domain of the function, but placed at its periphery have a much different weight than those in the vicinity of the point of interest.

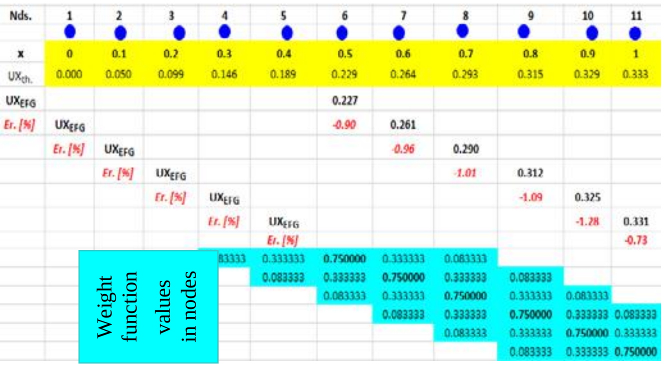

Figure 19. Johnson quartic function, $d_m = 1.5$

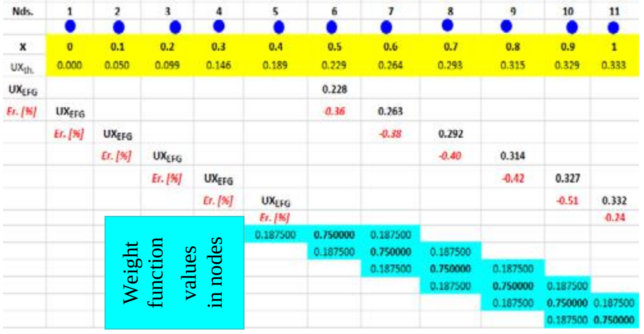

Figure 20. Johnson quartic function, $d_m = 1.0$

The scientific research carried out, part of which is presented in this paper, as well as our practical experience recommend the use of this weighting function for the numerical simulation of some phenomena with strong local specificity.

C. EFG Solution of the Problem

For this case study, the solution found by numerical solving the problem is presented in Table II, A), B) and Fig. 21, compared to the analytical solution. A program made in Matlab was used, using the same number of nodes as in the previous tests.

For any of the numerical methods used, the errors of the numerical results compared to the (exact) analytical results can be evaluated quantitatively, both as a percentage and by a norm (*e_norm*), defined by the expression:

$$e\_norm = \frac{1}{N}\sum_{j=1}^{N} \frac{\left| u\left(x_j\right)^{num} - u\left(x_j\right)^{eact} \right|}{\left| u\left(x_j\right)^{eact} \right|} \qquad (21)$$

The evaluation of the precision of the numerical solution compared to the analytical solution can also be done by evaluating the energy norm, with the relation:

$$energy\ \ norm = \left[\frac{1}{2} \int_0^1 \left(u_{,x}^{num} - u_{,x}^{exact} \right)^T \cdot E \cdot \left(u_{,x}^{num} - u_{,x}^{exact} \right) \cdot dx \right]^{0.5} \qquad (22)$$

TABLE II
A) VALUES OF ANALYTICAL AND NUMERICAL SOLUTIONS (EFG) AND ERRORS

x	$u_{teoretic}$	u_{EFG}^{cubic}	err. [%]	e_norm	energy norm
0	0	0	0	**0.003128**	**0.006252**
0.1	0.049833	0.050235	0.80		
0.2	0.098666	0.098828	0.16		
0.3	0.145500	0.145904	0.28		
0.4	0.189333	0.189832	0.26		
0.5	0.229166	0.229799	0.28		
0.6	0.264000	0.264748	0.28		
0.7	0.292833	0.293745	0.31		
0.8	0.314666	0.315528	0.27		
0.9	0.328500	0.330229	0.52		
1.0	0.333333	0.332470	0.26		

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B) VALUES OF ANALYTICAL AND NUMERICAL SOLUTIONS (EFG) AND ERRORS

x	$u_{teoretic}$	$u_{EFG}^{quartic}$	err. [%]	e_norm	energy norm
0	0	0	0	**0.005714**	**0.011309**
0.1	0.049833	0.050982	2.31		
0.2	0.098666	0.098556	-0.11		
0.3	0.145500	0.146310	0.56		
0.4	0.189333	0.190034	0.37		
0.5	0.229166	0.230193	0.45		
0.6	0.264000	0.265090	0.41		
0.7	0.292833	0.294398	0.53		
0.8	0.314666	0.315655	0.31		
0.9	0.328500	0.331651	0.96		
1.0	0.333333	0.332425	-0.27		

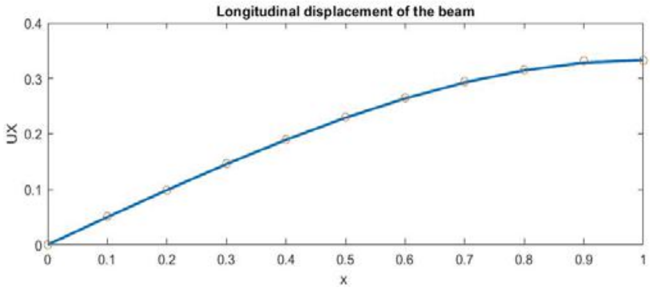

Figure 21. Curve of nodal displacement for quartic weight function

IV. TIMOSHENKO BAR STUDY WITH EFG METHOD

The study of the Timoshenko bar is often invoked in the literature as a calculus validation test. Keeping this tradition, we still present, in a synthetic way, the results of the calculation with the EFG method, using a software for 2D structure, made in Matlab. In our numerical study (Fig. 22), we used the following numerical data: $L = 0.48\ m$, $D = 0.12\ m$, $b = 0.10\ m$, $E = 3 \cdot 10^7\ Pa$, $\nu = 0.30$ and $P = 1000\ N$. For (second) moment of inertia $I_y \equiv I$, the resulting value is $1.44 \cdot 10^{-5}\ m^4$.

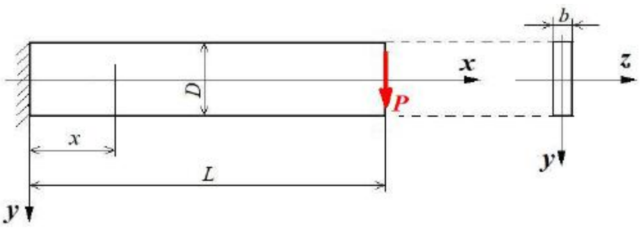

Figure 22. The model of Timoshenko bar

The displacement solution of this problem is known from the literature [12-14]. Nodal displacements along y-axis, for any point, can be calculated with (21)

$$u_y = \frac{P}{6EI}\left[3\nu y^2 (L-x) + (4+5\nu)\frac{D^2 x}{4} + (3L-x)x^2\right] \quad (23)$$

For the points on the longitudinal axis of the bar, the transverse displacements (along the y-axis) are calculated with (24), which results from (23)

$$u_y\big|_{y=0} = \frac{P}{6EI}\left[(4+5\nu)\frac{D^2 x}{4} + (3L-x)x^2\right] \quad (24)$$

Using (24) for the free end of the bar leads to the result presented by (25)

$$u_{y_{\max}}\Big|_{y=0}^{theoretic} = 0.088999\ m \quad (25)$$

We present in this paper, a part of the results, regarding weighting functions, values of the dimensional parameters of the support domain like d_m, c_I, d_{nn}, d_{mI}, described in subchapter B and graphical presented in Fig. 2. Of course, several variants of the number of field nodes were used for the considered structure.

The results of the numerical calculation are compared with the theoretical values by evaluating the percentage error. Accuracy is also assessed through the energy norm, calculated by the program.

The results of the numerical study with the EFG method are summarized in Table III, A), B) where the result accuracy is evaluated as a percentage and by the energy norm.

TABLE III, A) EFG RESULTS FOR TIMOSHENKO BEAM

		Cubic spline weight function		
		EFG values		***Err. [%]***
d_m	1.5	$u_{y_{\max}}$	**0.088897**	***-0.11***
c_I	0.01			
d_{nn}	0.01	*energy norm*	*0.6684*	-
d_{mI}	0.015			
d_m	1.5	$u_{y_{\max}}$	**0.088543**	***-0.51***
c_I	0.02			
d_{nn}	0.02	*energy norm*	*1.4269*	-
d_{mI}	0.03			
d_m	2.00	$u_{y_{\max}}$	**0.088991**	***-0.009***
c_I	0.01			
d_{nn}	0.01	*energy norm*	*0.1677*	-
d_{mI}	0.02			
d_m	2.00	$u_{y_{\max}}$	**0.088543**	***-0.51***
c_I	0.02			
d_{nn}	0.02	*energy norm*	*0.4602*	-
d_{mI}	0.04			
d_m	3.00	$u_{y_{\max}}$	**0.088990**	***-0.009***
c_I	0.01			
d_{nn}	0.01	*energy norm*	*0.0642*	-
d_{mI}	0.03			
d_m	3.00	$u_{y_{\max}}$	**0.089008**	***0.01***
c_I	0.02			
d_{nn}	0.02	*energy norm*	*0.1317*	-
d_{mI}	0.06			

Fig. 23 and 24, provided by the used program, show the geometric calculus model for the EFG method, respectively the deformed state of the bar, for the last variant presented in Table III, A), B), using the cubic spline weight function.

TABLE III, B) EFG RESULTS FOR TIMOSHENKO BEAM

		Quartic spline weight function		
		EFG values		Err. [%]
d_m	1.5	$u_{y\max}$	**0.088906**	***-0.10***
c_I	0.01			
d_{nn}	0.01	*energy norm*	*1.5058*	-
d_{mI}	0.015			
d_m	1.5	$u_{y\max}$	**0.088653**	***-0.39***
c_I	0.02			
d_{nn}	0.02	*energy norm*	*2.9367*	-
d_{mI}	0.03			
d_m	2.00	$u_{y\max}$	**0.094589**	***6.28***
c_I	0.01			
d_{nn}	0.01	*energy norm*	*5.3104*	-
d_{mI}	0.02			
d_m	2.00	$u_{y\max}$	**0.098977**	***11.21***
c_I	0.02			
d_{nn}	0.02	*energy norm*	*6.9827*	-
d_{mI}	0.04			
d_m	3.00	$u_{y\max}$	**0.124067**	***39.40***
c_I	0.01			
d_{nn}	0.01	*energy norm*	*13.3106*	-
d_{mI}	0.03			
d_m	3.00	$u_{y\max}$	**0.135171**	***51.88***
c_I	0.02			
d_{nn}	0.02	*energy norm*	*15.2588*	-
d_{mI}	0.06			

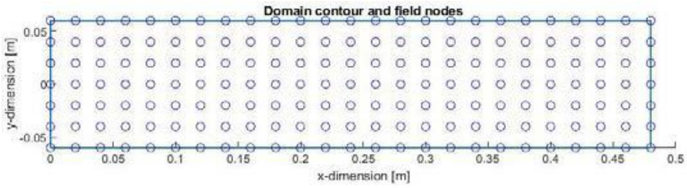

Figure 23. Calculus model by EFG method

Figure 24. Deformed state of the beam

V. CONCLUSIONS

The results of our study lead us to some conclusions and also answers to the questions asked in this paper which are presented below. Unfortunately, we cannot answer, unequivocally, the question of which weight function to use; the answer to this question can be found by the user by calibrating the method for the given problem.

Some key points, orientation towards an accurate answer can be formulated; these are presented below.

The most used weighting functions are the cubic spline weight function and the quartic weight function; these functions have been the most studied, comparatively, by us.

From the analysis of the allure of the weight function curves, we can appreciate that for the study of some phenomena with strong local specificity, the Johnson quartic weight functions and new quartic weight functions would be the most appropriate.

The weight function with least sensitive to its functional parameters is the cubic spline weight function; this observation (finding) explains its wide use; many professional calculus software using the EFG method have implemented this weight function [15], [16].

The quartic weight function leads to results very close to those of the cubic spline weight function, only for parameters that lead to small dimensions of the support domain; the size of the support domain for the point of interest (d_{mI}), for the positive values of the size of the influence domain of the respective point, must not exceed twice the internodal distance.

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